B. Math. I Year 2001-2002
I Semester Final Exam

Date: 21-11-2001 **Algebra I** **Max. Marks:**

Instructions: All questions, even if they are not of the same level of difficulty, carry equal marks.

1. Determine the centralizer of a k-cycle in the permutation group S_n, where $1 \leq k \leq n$.

2. Show that there exists a unique non-abelian group of order $2p$ where p is an odd prime.

3. Let G be a group of order $p^e \cdot m$ where $(p, m) = 1$ and p is a prime. Show that G contains a subgroup of order p^e. (Sylow's first theorem.)

4. Let G be a group of order $p^e \cdot a$ with $1 \leq a < p; p$ prime and $e \geq 1$. Show that G is not simple.

5. If H is a subgroup of G, define $N(H) = \{g \in G | gHg^{-1} = H\}$. If P is a p-Sylow subgroup of G, prove that $N(N(P)) = N(P)$.

6. Let H, N be normal subgroups of G such that $N \subset H$. Let $\bar{G} = G/N$ and $\bar{H} = H/N$. Prove that $\bar{H}$ is a normal subgroup of $\bar{G}$ and G/H is isomorphic to $\bar{G}/\bar{H}$. (The third isomorphism theorem.)

7. Prove that a square matrix is invertible if and only if its columns are linearly independent.

8. (a) Let W be a subspace of a finite dimensional vector space V. Show that there is a subspace W_1 of V such that $V = W \oplus W_1$. Is W_1 unique?

   (b) Let v be a non-zero vector in a finite dimensional vector space V over a field F. Show that there is a linear transformation $T : V \to F$ such that $T(v) \neq 0$.

9. A linear transformation $T : V \to V$ is called nilpotent if $T^k \equiv 0$ for some k. If dimension of V is n, show that $T^n \equiv 0$ for any nilpotent $T : V \to V$. (Hint: Prove that $Im(T^{i+1})$ is a proper subspace of $Im(T^i)$ if $Im(T^i) \neq \{0\}$.)